Final Examination: Stochastic Calculus and Applications
Fall Term 2003.

Instructions. You may consult any books or articles that you find useful. You may use any software. You may **NOT** consult with any other person, except on the essay question. You should strive to make your answers as **clear and complete** as possible — neatness counts. If you find a bug in a problem, modify the problem in a way that speaks to the highest intellectual aspiration that the problem might reasonably suggest — then solve the modification.

Your Options. You can *either* (1) do the suggested problems and do the short essay (4-6 pages) *or* (2) you can just do the longer essay (8-12 pages).

Part One: Problems

Problem 1.

Consider the processes defined by *definite* integral

$$X_t = \sqrt{2}\ t^2 \int_0^\infty B_u e^{-ut}\, du$$

and the *definite* stochastic integral

$$Y_t = \sqrt{2}\ t \int_0^\infty e^{-ut}\, dB_u.$$

(a) Calculate $E(X_t)$, $E(X_t^2)$, $E(Y_t)$, and $E(Y_t^2)$.

(b) Show that the processes $\{X_t\}$ and $\{Y_t\}$ are actually equivalent. That is, show that for any $0 \leq t_1 < t_2 < \cdots t_n$ the vector $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ has the same distribution as the vector $(Y_{t_1}, Y_{t_2}, ..., Y_{t_n})$.

(c) Show that the processes $Z_t = X_{at}$ and $Z'_t = \sqrt{a}X_t$ are equivalent. Do you know any other process with this scaling property?

(d) Show that the process defined by $W_0 = 0$ and $W_t = tX_{1/t}$ for $t > 0$ is equivalent to the process $\{X_t\}$. Do you know any other process with this scaling property?

PROBLEM 2.

In our examination of the binomial arbitrage, we found considerable convenience in the assumption that the interest rate was zero. Very pleasantly, the nature of self-financing strategies is such that we can almost always restrict attention to this case.

Suppose that the processes $\{S_t\}$ and $\{\beta_t\}$ satisfy the equations of the SDEs

$$dS_t = \mu(\omega, t)\, dt + \sigma(\omega, t)\, dB_t \quad \text{and} \quad d\beta_t = r\beta_t\, dt. \tag{1}$$

If the trading strategy (a_t, b_t) is self-financing for the processes S_t and β_t in the sense that it satisfies

$$a_t S_t + b_t \beta_t = a_0 S_0 + b_0 \beta_0 + \int_0^t a_t\, dS_t + \int_0^t b_t\, d\beta_t, \tag{2}$$

show that for any measurable adapted $\gamma_t > 0$, we also have

$$a_t(\gamma_t S_t) + b_t(\gamma_t \beta_t) = a_0(\gamma_0 S_0) + b_0(\gamma_0 \beta_0) + \int_0^t a_t\, d(\gamma_t S_t) + \int_0^t b_t\, d(\gamma_t \beta_t), \tag{3}$$

provided that all the indicated integrals are well defined.

PROBLEM 3. In general, the portfolio weights (a_t, b_t) of martingale pricing theory are given by the familiar formulas

$$a_t = \frac{u(\omega, t)}{d(\omega, t)} \quad \text{and} \quad b_t = U_t - \frac{u(\omega, t)}{d(\omega, t)} D_t. \tag{4}$$

On the other hand, the PDE method can be applied under the classic Black–Scholes model with constant μ, σ, and r to show that the for the call option with strike price K the replication portfolio has the concrete weights

$$a_t = f_x(t, S_t) \quad \text{and} \quad b_t = \frac{1}{r\beta_t}\left\{ f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t)\sigma^2 S_t^2 \right\}, \tag{5}$$

where $f(t, x)$ is given by the Black–Scholes formula.

Show that the concrete weights (5) may be obtained from the more abstract weights by exploiting the fact that we have two representations for the arbitrage price of the call option:

$$f(t, S_t) = \beta_t E_Q((S_T - K)_+ / \beta_T | \mathcal{F}_t) = \beta_t U_t.$$

PROBLEM 4. Consider the process $X_t = ||(B_t^{(1)}, B_t^{(2)}, \ldots, B_t^{(d)})||$ where $B_t^{(k)}$, $k = 1, 2, ..., d$ are independent Brownian motions and $|| \cdot ||$ is just the Euclidean norm.

(a) Find an SDE for X_t of the form

$$dX_t = \mu_t dt + \sigma_t dB_t$$

where B_t is a standard Brownian motion. Here μ_t and σ_t should be given as functions of X_t and (possibly) t, since the process X_t is Markovian.

(b) Suggest a generalization of the SDE of part (a). Note that the process X_t of part (a) is nonnegative for all t. Do you believe that your SDE will also have nonnegative solutions? A clear intuitive discussion is all that is required here, but if you can provide an honest proof that would be excellent.

ESSAY QUESTIONS

If you take option (1) you need to solve the preceding question and you also need to write a four to six page essay on material related to our course. If you take option (2) you only have to write an essay, but it must be a more substantial one of six to ten pages. In each case there are constraints on style and form.

CONSTRAINTS ON STYLE AND FORM

1. The essay must be done using latex.
2. The essay should be done using the article style template.
3. Your essay must have an abstract, an introduction, a conclusion, a bibliography, and material in the middle that is worth reading.
4. Your essay must be carefully prepared with attention to proper grammar, spelling, notation, and mathematical display. You should read your essay *many times* to make sure that it is edited carefully. Here it is perfectly acceptable to ask a friend to read your essay and to provide advice.

CONSTRAINTS ON CONTENT

1. The material must be original. Any sources which are drawn upon must be properly cited.
2. The material must be *tightly* related to this course, though naturally it can reflect your other interests.
3. The material must be understandable as a self-standing piece. If results that are not part of this course are used, they must be stated completely.
4. You essay will be judged as much for comprehensibility as for any other quality.

Advice on Content and Style

1. Those taking option (1) should not be afraid of aiming for a very modest objective. For example, if you can find a problem in a book, solve that problem, think of some modest variations on that problem, and outline your solutions for those, you will have a fine essay.

2. Those taking option (2) are expected to do more than solve some text book problems, but it still pays to keep the objective reasonably modest. If you have a model that is of interest to you, ask the questions that would be natural from the context of our course, and do your best. Even negative results can be interesting if well written, and you should keep in mind that most ideas do not work.

   If your essay has substantial overlap with work that you are doing for another class, we should discuss the steps you should take to avoid inappropriate double counting.

3. The best essays are likely to come from questions that you have asked yourself during the course and which you managed to solve (or make concrete progress toward solving).