Bayesian Inference for Derivative Prices

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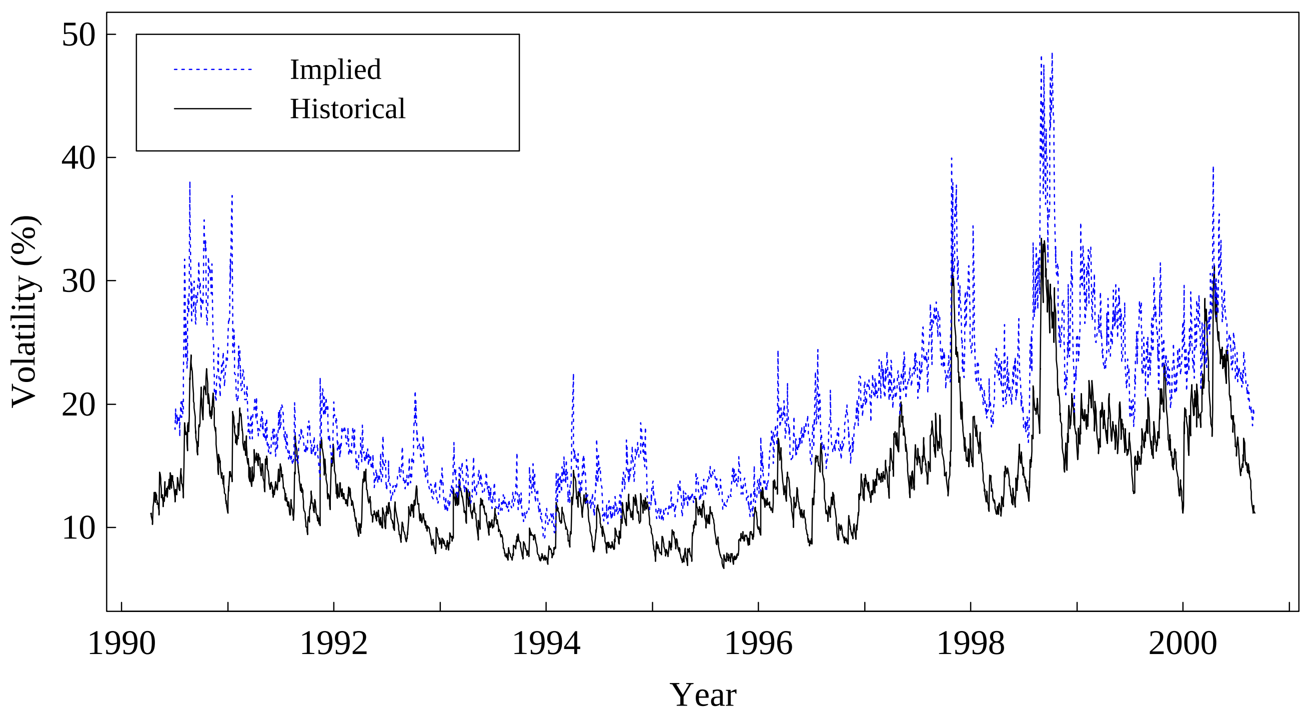
Overview

- A Volatility Puzzle
- Bayesian Inference (Derivative and Asset Prices)
- Advantages of Joint Inference
- Application: S&P500 Option Pricing
- Discussion

A Volatility Puzzle

- Option prices quoted as Black-Scholes implied volatilities (IVs)
  IVs typically higher than estimated volatilities.
- Skew and Smile to implied volatilities due to Stochastic Volatility, Leverage and Jumps (e.g, Bates 1996, EJP 2002).
  Market price of volatility risk λ_v



Bayesian Inference

- Data $Y = (Y^D, Y^S)$ both Derivative and Asset price information.
  State variables $X = \{X_t\}_{t=1}^T$ and Parameters Θ, Λ
- Asset Prices $\quad p(X, \Theta | Y^S) \propto p(Y^S | X, \Theta) p(X, \Theta)$
- Derivative prices

$$p(X, \Theta | Y) \propto p(Y^D | Y^S, X, \Theta) p(X, \Theta | Y^S)$$

  where

$$Y_t^D = f(Y_t^S, X_t, \Theta, \Lambda) + \epsilon_t^D$$

- Pricing Error ϵ_t^D avoids stochastic singularities in state variables.
- Joint Inference

$$p(X, \Theta, \Lambda | Y) \propto p(Y^D | Y^S, X, \Theta, \Lambda) p(Y^S | X, \Theta) p(X, \Theta) p(\Lambda)$$

  See JP (2002) and PSM (2002) for interest rate applications

Advantages

- Can extract state and parameters with marginal posteriors

$$p(X|Y) \text{ and } p(\Theta|Y) \text{ and } p(\Lambda|Y)$$

  Hard to do this with simulated method of moments (CG 2002)

- Sharper Parameter and State estimates

- Model Misspecification Diagnostic
  - Comparison of time series properties of $\{\hat{X}_t\}_{t=0}^T$ (estimated with and without derivatives) indicates model mispricing
  - Disagreement of $p(\Theta|Y^D, Y^S)$ and $p(\Theta|Y^S)$ also indicates model misspecification

Stochastic Volatility Option Pricing

- Heston (1993) considers pricing a call option with stochastic volatility and leverage:

$$Y_t^D = E^Q\left(e^{-r\tau}(S_T - K)_+ | V_t, \Theta, \Lambda\right)$$

- Risk-neutral dynamics (E^Q)

$$\frac{dS_t}{S_t} = r dt + \sqrt{V_t}\, dW_t^s$$
$$dV_t = \kappa^*\left(\theta^* - V_t\right) + \sigma_v \sqrt{V_t}\, dW_t^v$$

- Leverage effect $\rho = \text{corr}\left(dW_t^s, dW_t^v\right)$ (negative)
- Market price of risk λ_v adjusts drift $(\kappa, \theta) \to (\kappa^*, \theta^*)$:

$$\kappa^* = \kappa + \lambda_v \quad \text{and} \quad \theta^* = \frac{\kappa}{\kappa + \lambda_v}\theta$$

A "Closed-Form" Solution

Call price formula:

$$Y_t^D = S_t P_1(V_t, \Theta) - K e^{-r\tau} P_2(V_t, \Theta)$$

Probabilities P_j given by Fourier inversion:

$$P_j(V_t, \Theta) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} Re \left[\frac{e^{-i\phi \ln(K)} f_j(V_t, \Theta)}{i\phi} \right] d\phi$$

where

$$f_j(V_t, \Theta) = e^{C(\tau;\phi) + D(\tau;\phi)V_t + i\phi \ln(S_t)}$$

DPS (2000) show how to include jumps.

EJP (2002) estimate these models.

MCMC Implementation

Full Conditionals:

1. Asset price drift:	$p(\mu \mid \text{rest})$	(Gibbs)
2. Pricing variance:	$p(\sigma_D^2 \mid \text{rest})$	(Gibbs)
3. Volatility states:	$p(\{V_t\}_{t=1}^T \mid \text{rest})$	(Single-state Metropolis)
4. Volatility drift:	$p(\kappa, \theta \mid \text{rest})$	(Random Walk Metropolis)
5. Leverage, vol of vol:	$p(\rho, \sigma_v \mid \text{rest})$	(Random Walk Metropolis)
6. Market price of risk:	$p(\lambda_v \mid \text{rest})$	(Random Walk Metropolis)

Application: S&P 500 Equity Options

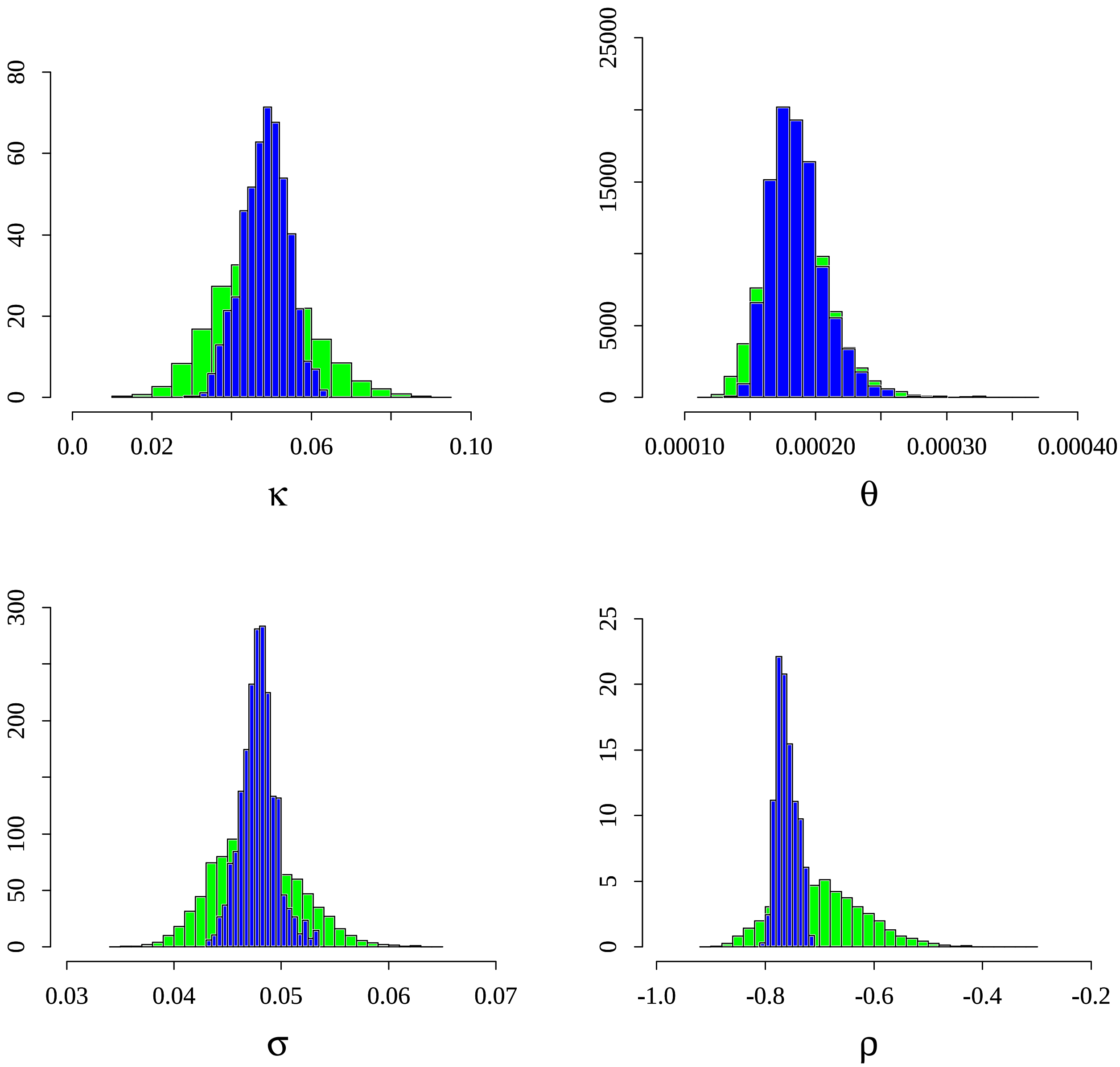
- $T = 923$ trading days (1998–2002)
- Asset data: S&P 500 index returns
- Derivative data: Panel of B-S implied volatilities at $\Delta = \{0.25, 0.5, 0.75\}$ with expiration $\tau = 1$ month.

- Figure 1: Sharper Posteriors for $\Theta = (\kappa, \theta, \rho, \sigma_v)$
- Figure 2: Posteriors for λ_v and $\frac{\kappa}{\kappa+\lambda_v}$
- Figure 3: Smoothed Volatilities V_t.
- Figure 4: Posteriors for $\Theta = (\kappa, \theta, \sigma_v)$ with no leverage.

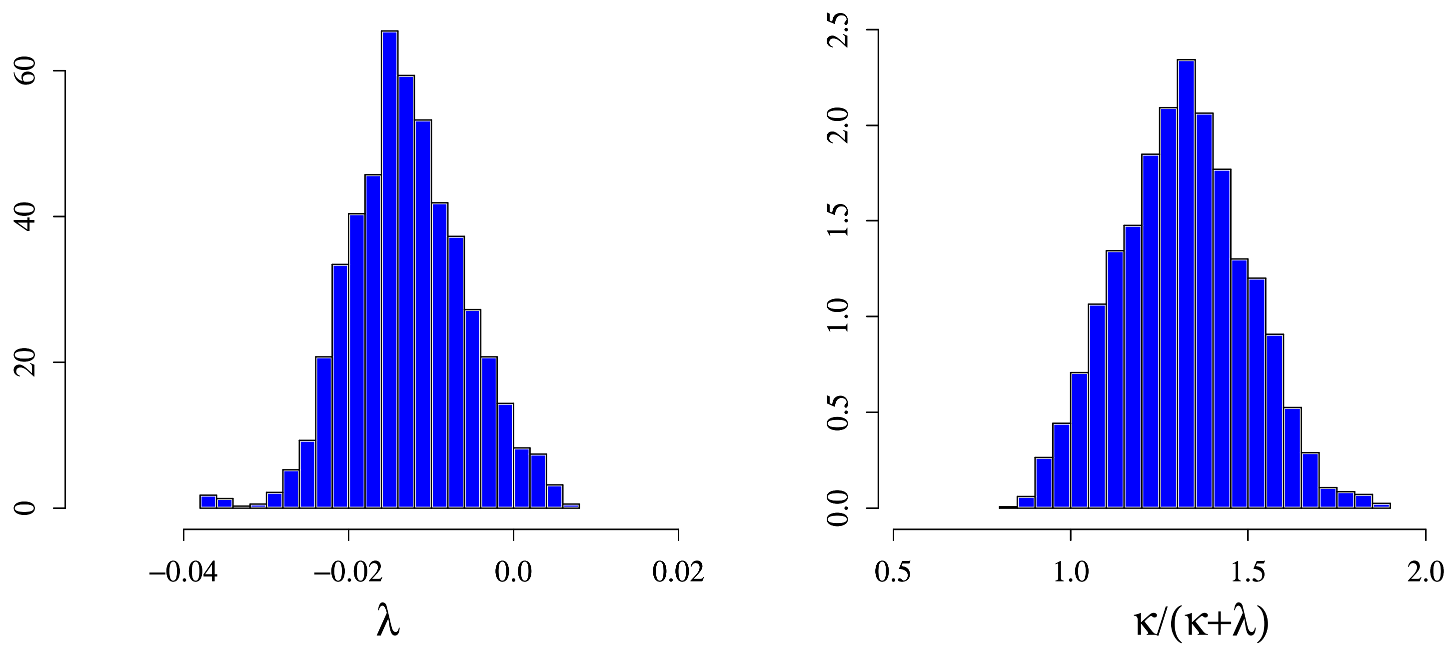
Posterior Estimates (Annualized)

Parameter	Y^S only	Y^S and Y^D	
		$\sigma_D = 5\%$	σ_D unknown
$E(\sqrt{V_t})$	21.60%	21.90%	21.82%
	(1.46)	(1.15)	(0.86)
κ	0.045	0.046	0.055
	(0.012)	(0.005)	(0.005)
σ_v	0.047	0.048	0.055
	(0.004)	(0.002)	(0.001)
ρ	-0.70	-0.78	-0.74
	(0.08)	(0.02)	(0.01)
λ_v	–	-0.010	-0.017
	–	(0.005)	(0.005)

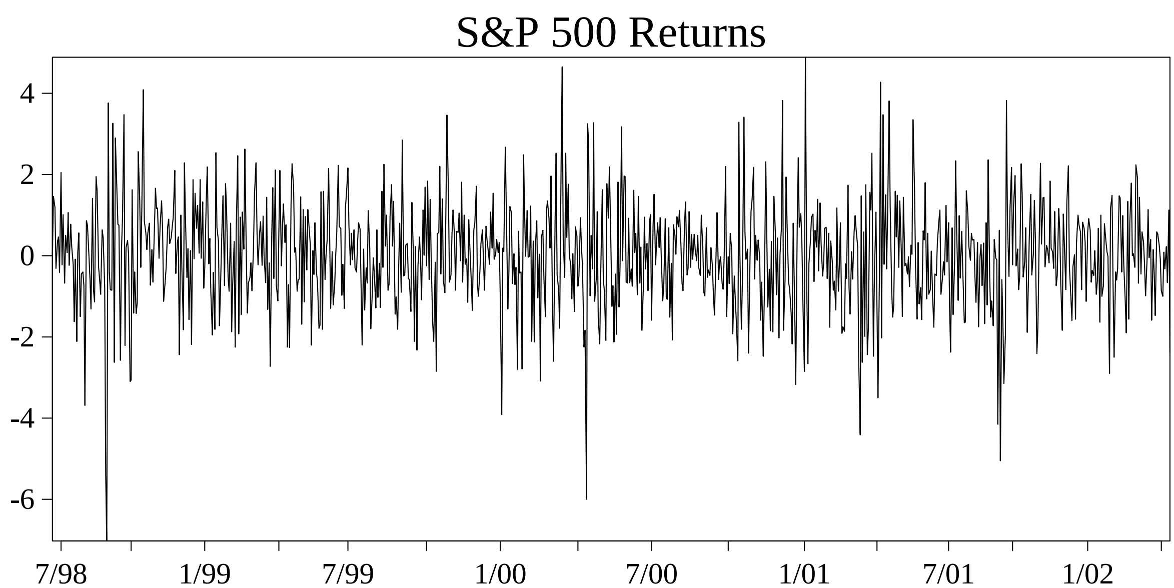
Posteriors with and without Derivative Prices

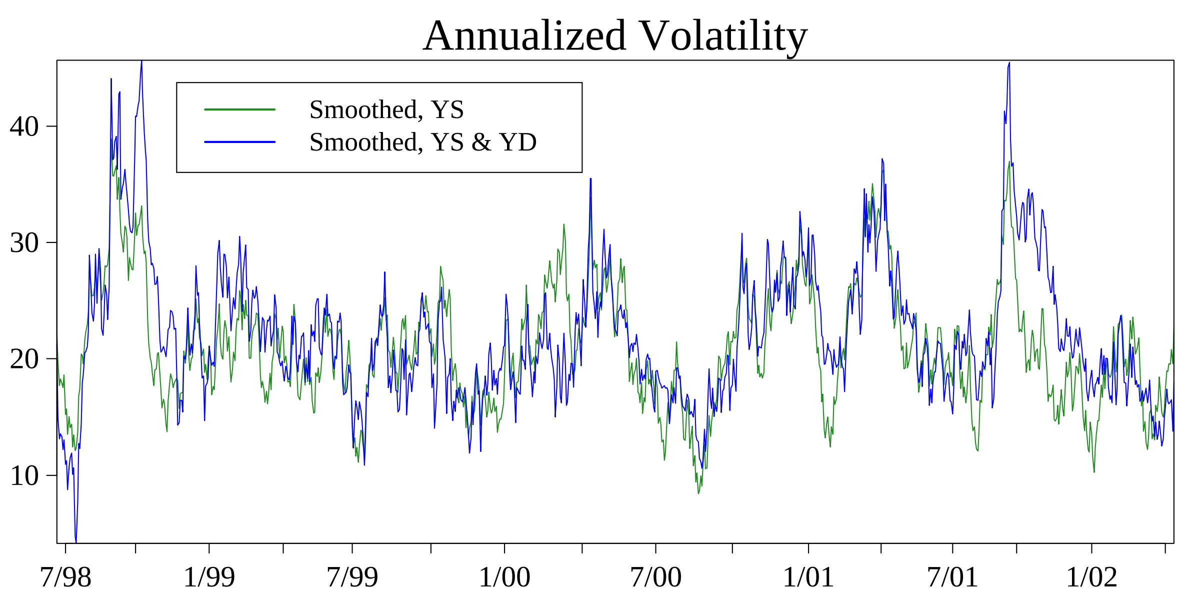


Market Price of Risk Parameters

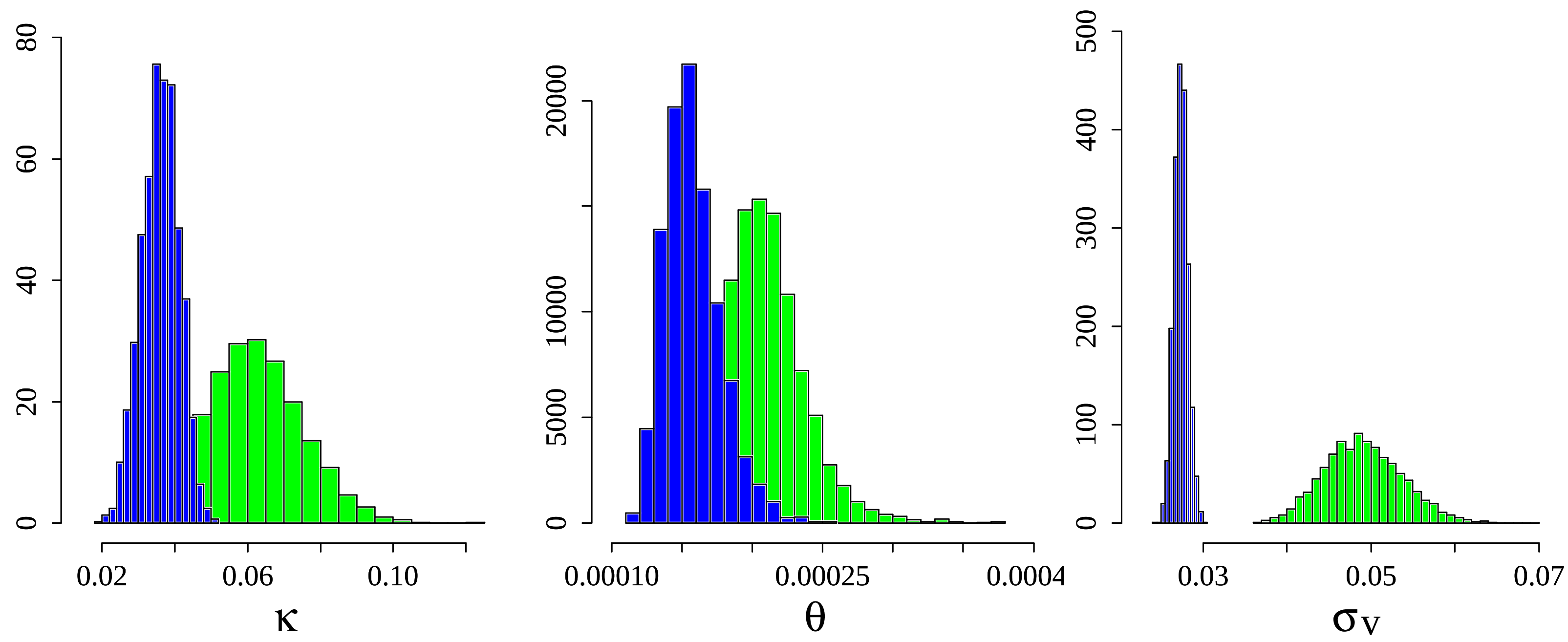


Returns and Smoothed Volatilities





Posteriors with and without Derivative Prices
No Leverage ($\rho = 0$)



Model Misspecification

- Example 1: Figure 3 shows the estimated state variables $\hat{V}_t$ with and without derivative prices.

  There are a number of periods of significant disagreement, e.g. Aug 26–Oct 20th, 1998 (Russian debt and LTCM crisis)

- Example 2: Assume that leverage effect is zero ($\rho = 0$)

  Figure 5 compares the posterior distributions $p(\Theta|Y^D, Y^S)$ and $p(\Theta|Y^S)$

  Posteriors distributions on $(\kappa, \theta, \sigma_v)$ are dramatically different under the two procedures.

Discussion

- Joint Inference for Derivative and Asset Prices
- Sharper Parameter and State Estimation
- Model Misspecification Diagnostics

  Check disagreement between state variables and parameters

  Indicates a jump state variable
- Further issues: Sequential Parameter Learning + Pricing and Assessing Estimation Risk.